Ring Theory 10

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Q) R is a polynomial ring of one variable x over the field $\mathbb{Z}/5\mathbb{Z}$. Multiplicative set $M = \{1, x+1, (x+1)^2, \dots\}$. Localization at M.

$\{1, x, x^2 \dots\}$ $\to M^{-1}R = \mathbb{Z}/5\mathbb{Z}[x, \frac{1}{x}]$

Ans:— $M^{-1}R = \mathbb{Z}/5\mathbb{Z}[x, \frac{1}{x+1}]$

$R = 2\mathbb{Z}$, $S = 2\mathbb{Z} - \{0\}$.

$S^{-1} \longrightarrow \frac{1}{s}$ $\quad S^{-1}R = \{\frac{r}{s}, r \in 2\mathbb{Z}, s \in 2\mathbb{Z} - \{0\}\}$

$\frac{p}{q} \in \mathbb{Q}$ $\quad p \in \mathbb{Z}$, $2p \in 2\mathbb{Z}$, $2q \in 2\mathbb{Z} - \{0\}$ $\Rightarrow \frac{2p}{2q} \in S^{-1}R \Rightarrow \frac{p}{q} \in S^{-1}R$

Lemma:— Let $\varphi : R \to S$ be ring map. If R and S are local rings then the following are equivalent.

(i) φ is a local ring map

(ii) $\varphi(m_R) \subset m_S$

(iii) $\varphi^{-1}(m_S) = m_R$

(iv) For any $x \in R$, if $\varphi(x)$ is invertible in S then x is invertible in R

Proof:— (i) & (ii) are equiv. by definition.

(iii) $\Rightarrow$ (ii)

$\varphi^{-1}(m_S)$ is a prime ideal containing the maximal ideal m_R.

$\Rightarrow \varphi^{-1}(m_S) = m_R$ $\quad \Rightarrow$ (ii) $\Rightarrow$ (iii) $\quad \Rightarrow$ (ii) $\Leftrightarrow$ (iii)

$\varphi(m_R) \subset m_S$
$x \in R$ and $\varphi(x)$ is invertible in S $\Rightarrow \varphi^{-1}(\varphi(x))$ is invertible in R
$\Rightarrow$ x is invertible in R
$\Rightarrow$ (ii) $\Leftrightarrow$ (iv)

Lemma:— Let R be a ring. The following are equivalent

(i) R is a local ring

(ii) R has a maximal ideal m and every element of $R \setminus m$ is a unit

(iii) R is not the zero ring and for every $x \in R$ either x or $1-x$ is invertible or both

Proof:- (i) & (ii) are equiv by definition

Let R be local.

Then $x \in R$ we have x either in m or not in m $\Rightarrow$ (i) $\Rightarrow$ (ii)

Let (iii) be true, then,

Let m, m' be distinct maximal ideals. Let $x \in R$ be such that $x \bmod m' = 0$ and $x \bmod m = 1$ (by Chinese Remainder Theorem)

x is not invertible in m' and $1-x$ is not invertible in m

$\Rightarrow$ x and $1-x$ both are not invertible in R $\Rightarrow\Leftarrow$

$\Rightarrow m = m'$ $\qquad \Rightarrow$ (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

The localization R_p of a ring R at a prime p is a local ring with maximal ideal pR_p